

κ -METRIZABLE COMPACTA AND SUPEREXTENSIONS

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ABSTRACT. A characterization of κ -metrizable compacta in terms of extension of functions and usco retractions into superextensions is established.

1. INTRODUCTION

In this paper we assume that all topological spaces are compact Hausdorff and all single-valued maps are continuous.

Our main result is a characterization of κ -metrizable compacta (see Theorem 1.1 below) similar to the original definition of *Dugundji spaces* given by Pelczynski [12] (recall that the class of κ -metrizable spaces was introduced by Shchepin [17] and it contains Dugundji spaces). This characterization is based on a result of Shapiro [16, Theorem 3] that a compactum X is κ -metrizable if and only if for every embedding of X in a compactum Y there exists a *monotone extender* $u: C_+(X) \rightarrow C_+(Y)$, i.e. $u(f)|_X = f$ for each $f \in C_+(X)$ and $f \leq g$ implies $u(f) \leq u(g)$. Here, by $C(X)$ and $C_+(X)$ we denote all continuous, respectively, continuous and non-negative functions on X . Following [16], we say that a map $u: C(X) \rightarrow C(Y)$ is: (i) *monotone*; (ii) *homogeneous*; (iii) *weakly additive*, if for every $f, g \in C(X)$ and every real number k we have: (i) $u(f) \leq u(g)$ provided $f \leq g$; (ii) $u(k \cdot f) = k \cdot u(f)$; (iii) $u(f + k) = u(f) + k$. Properties (i), (ii) and (iii) imply that for every $f \in C(X)$ and $g \in C_+(X)$ the following hold: $u(1) = 1$, $u(0) = 0$ and $u(g) \geq 0$.

We also introduce and investigate the covariant functor S : $S(X)$ is the space of all monotone, homogeneous and weakly additive maps $\varphi: C(X) \rightarrow \mathbb{R}$ with the pointwise convergence topology. It is easily

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seen that the map $x \rightarrow \delta_x$, $x \in X$, where each δ_x is defined by $\delta_x(f) = f(x)$, $f \in C(X)$, is an embedding of X into $S(X)$. It appears that κ -metrizable compacta are exactly S -injective compacta in the sense of Shchepin [19]. The functor S , being a subfunctor of the functor O introduced by Radul [14], has nice properties. For example, S is open and weakly normal (see Section 2).

Here is the characterization of κ -metrizable compacta mentioned above.

Theorem 1.1. *For a compact space X the following conditions are equivalent:*

- (i) X is κ -metrizable;
- (ii) For any embedding of X in a compactum Y there exists a monotone, homogeneous and weakly additive extender $u: C(X) \rightarrow C(Y)$;
- (iii) For any embedding of X in a compactum Y there exists a continuous map $r: Y \rightarrow S(X)$ such that $r(x) = \delta_x$ for all $x \in X$.
- (iv) For any embedding of X in a compactum Y there exists an usco map $r: Y \rightarrow \lambda X$ such that $r(x) = \eta_x$ for all $x \in X$;

Theorem 1.1 remains valid if S is replaced by the functor O , see Theorem 4.2.

Recall that supercompact spaces and superextensions were introduced by de Groot [5]. A space is *supercompact* if it possesses a binary subbase for its closed subsets. Here, a collection \mathcal{S} of closed subsets of X is *binary* provided any linked subfamily of \mathcal{S} has a non-empty intersection (we say that a system of subsets of X is *linked* provided any two elements of this system intersect). The supercompact spaces with binary *normal* subbase will be of special interest for us. A subbase \mathcal{S} is called *normal* if for every $S_0, S_1 \in \mathcal{S}$ with $S_0 \cap S_1 = \emptyset$ there exists $T_0, T_1 \in \mathcal{S}$ such that $S_0 \cap T_1 = \emptyset = T_0 \cap S_1$ and $T_0 \cup T_1 = X$. A space X possessing a binary normal subbase \mathcal{S} is called *normally supercompact* [10] and will be denoted by (X, \mathcal{S}) .

The *superextension* λX of X consists of all maximal linked systems of closed sets in X . The family

$$U^+ = \{\eta \in \lambda X : F \subset U \text{ for some } F \in \eta\},$$

$U \subset X$ is open, is a subbase for the topology of λX . It is well known that λX is normally supercompact. Let η_x , $x \in X$, be the maximal linked system of all closed sets in X containing x . The map $x \rightarrow \eta_x$ embeds X into λX . The book of van Mill [10] contains more information about normally supercompact space and superextensions, see also Fedorchuk-Filippov's book [4].

The paper is organized as follows. Section 2 is devoted to the functor S . In Section 3 we investigate the connection between \varkappa -metrizable compacta and spaces with a closed binary subbase. As corollaries we obtain Ivanov's results from [7] and [8] concerning superextensions of \varkappa -metrizable compacta, as well as, results of Moiseev [11] about closed hyperspaces of inclusions. The proof of Theorem 1.1 is established in the final Section 4.

2. MONOTONE, HOMOGENEOUS AND WEAKLY ADDITIVE FUNCTIONALS

The space $S(X)$ is a subset of the product space $\mathbb{R}^{C(X)}$ identifying each $\varphi \in S(X)$ with $(\varphi(f))_{f \in C(X)} \in \mathbb{R}^{C(X)}$. Let us also note that, according to [14, Lemma 1], every monotone weakly additive functional is a non-expanding. So, all $\mu \in S(X)$ are continuous maps on $C(X)$.

Proposition 2.1. *Let X be a compact space. Then $S(X)$ is a compact convex subset of $\mathbb{R}^{C(X)}$ containing the space $P(X)$ of all regular probability measures on X .*

Proof. It can be established by standard arguments that $S(X)$ is a convex compact subset of $\mathbb{R}^{C(X)}$. Since $P(X)$ is the space of all monotone linear maps $\mu: C(X) \rightarrow \mathbb{R}$ with $\mu(1) = 1$, equipped with the pointwise topology, $P(X)$ is a subspace of $S(X)$. \square

Obviously, for any map $f: X \rightarrow Y$ between compact spaces the formula $(S(f)(\mu))(h) = \mu(h \circ f)$, where $\mu \in S(X)$ and $h \in C(Y)$, defines a map $S(f): S(X) \rightarrow S(Y)$. Moreover, $S(g \circ f) = S(g) \circ S(f)$. So, S is covariant functor in the category COMP of compact spaces and (continuous) maps.

We say that a covariant functor F in the category COMP is *weakly normal* if it satisfies the following properties:

- (a) F preserves injective maps (F is monomorphic);
- (b) F preserves surjective maps (F is epimorphic);
- (c) F preserves intersections ($F(\bigcap_{\alpha \in A} X_\alpha) = \bigcap_{\alpha \in A} F(X_\alpha)$ for any family of closed subsets X_α of a given compactum X);
- (d) F is continuous (F preserves limits of inverse systems);
- (e) F preserves weight of infinite compacta;
- (f) F preserves points and the empty set.

The above properties were considered by Shchepin [19] who introduced the import notion of a *normal functor*: a weakly normal functor preserving preimages of sets. We already mentioned that the functor

S is subfunctor of the functor O of order preserving functionals introduced by Radul [14]. Since O is weakly normal [14], we conclude that S is injective, preserves weight, points and the empty set.

Theorem 2.2. *The functor S is weakly normal, but not normal.*

Proof. We follow the corresponding proof from [14] for the functor O . First, let us show S is surjective. Suppose $f: X \rightarrow Y$ is a surjective map between compacta and $\nu \in S(Y)$. Denote by A the subspace of $C(X)$ consisting of all functions $h \circ f$, $h \in C(Y)$. A has the following properties: $0_X \in A$ and A contains both $\varphi + k$ and $k\varphi$ for any constant $k \in \mathbb{R}$ provided $\varphi \in A$ (such A will be called a *weakly additive homogeneous subspace* of $C(X)$). We define a monotone, weakly additive and homogeneous functional $\mu': A \rightarrow \mathbb{R}$ by $\mu'(f \circ h) = \nu(h)$. If μ' can be extended to a functional $\mu \in S(X)$, then $S(f)(\mu) = \nu$. So, next claim completes the proof that S is epimorphic.

Claim. *Let $\mu': A \rightarrow \mathbb{R}$ be a monotone, weakly additive and homogeneous functional. Then μ' can be extended to a monotone, weakly additive and homogeneous functional $\mu: C(X) \rightarrow \mathbb{R}$.*

Following the proof of [14, Lemma 2], we consider the pairs (B, μ_B) , where $B \supset A$ and μ_B are, respectively, a weakly additive homogeneous subspace of $C(X)$ and a monotone weakly additive homogeneous functional on B . We introduced a partial order on these pairs by $(B, \mu_B) \leq (C, \mu_C)$ iff $B \subset C$ and μ_C extends μ_B . Then, according to Zorn Lemma, there exists a maximal pair (B_0, μ_0) . If $B_0 \neq C(X)$, take any $\varphi_0 \in C(X) \setminus B_0$ and let B_0^+ (resp., B_0^-) to be the set of all $\varphi \in B_0$ with $\varphi \geq \varphi_0$ (resp., $\varphi \leq \varphi_0$). Because μ_0 is monotone, there exists $p \in \mathbb{R}$ such that $\mu_0(B_0^-) \leq p \leq \mu_0(B_0^+)$. Then B_0 is disjoint with the set $\{k\varphi + c : k, c \in \mathbb{R}\}$ and $D = B_0 \cup \{k\varphi + c : k, c \in \mathbb{R}\}$ is a weakly additive homogeneous subspace of $C(X)$. Define the functional $\mu: D \rightarrow \mathbb{R}$ by $\mu|_{B_0} = \mu_0$ and $\mu(k\varphi + c) = kp + c$ for all constants k, c . It is easily seen that μ is a monotone, weakly additive and homogeneous functional on D extending μ_0 . This contradicts the maximality of (B_0, μ_0) . Hence, $B_0 = C(X)$.

Therefore, S is epimorphic. Combining this fact with continuity of the functor O (see [14]), we obtain that S is also continuous. The arguments from the proof of [14, Lemma 5 and Proposition 5] imply that S preserves intersections. So, S is weakly normal. The example provided in [14] that O does not preserve preimages shows (without any modification) that S also does not preserve intersections. \square

Since S is monomorphic, for any $\mu \in S(X)$ the set $\text{supp}(\mu) = \bigcap \{H \subset X : H \text{ is closed and } \mu \in S(H)\}$ is well defined and is called support of

μ . Since S is continuous and epimorphic, next corollary follows from [19, Proposition 3.5].

Corollary 2.3. *The set $S_\omega(X) = \{\mu \in S(X) : \text{supp}(\mu) \text{ is finite}\}$ is dense in $S(X)$ for any compactum X .*

Theorem 2.4. *A surjective map $f: X \rightarrow Y$ is open if and only if the map $S(f): S(X) \rightarrow S(Y)$ is open.*

Proof. We say that a subset $A \subset S(X)$ is S -convex if A contains every $\mu \in S(X)$ with $\inf A \leq \mu \leq \sup A$. Here, the functional $\inf A: C(X) \rightarrow \mathbb{R}$ is defined by $(\inf A)(h) = \inf\{\nu(h) : \nu \in S(X)\}$, and similarly $\sup A$. Note that $\inf A$ and $\sup A$ are monotone and weakly additive, but not homogeneous. So, they are not elements of $S(X)$.

Now, using the notion of S -continuity one can proof this theorem following the arguments from the proof of [15, Theorem 1]. \square

Because κ -metrizable compacta are exactly the compacta which can be represented of the limit of an inverse sigma-spectrum with open bonding projections [19], we obtain the following corollary.

Corollary 2.5. *A compactum X is κ -metrizable if and only if $S(X)$ is κ -metrizable.*

Proposition 2.6. *For every compact space X there exists an embedding $i: \lambda X \rightarrow S(X)$ such that $i(\eta_x) = \delta_x$, $x \in X$. Moreover, $i(\lambda X) \neq S(X)$ provided X is disconnected.*

Proof. Following the proof of [16, Theorem 3], one can show that for every $\eta \in \lambda X$ and $f \in C(X)$ we have

$$(1) \quad \max_{F \in \eta} \min_{x \in F} f(x) = \min_{F \in \eta} \max_{x \in F} f(x)$$

and the equality $\varphi_\eta(f) = \max_{F \in \eta} \min_{x \in F} f(x)$ defines a monotone homogeneous and weakly additive map $\varphi_\eta: C(X) \rightarrow \mathbb{R}$. So, $\varphi_\eta \in S(X)$ for every $\eta \in \lambda X$. Moreover, $\varphi_{\eta_x}(f) = \max_{\{F: x \in F\}} \min_{y \in F} f(y) = f(x)$ for any $x \in X$. Therefore, we obtain a map $i: \lambda X \rightarrow S(X)$, $i(\eta) = \varphi_\eta$ with $i(\eta_x) = \delta_x$, $x \in X$. To prove the first part of the proposition we need to show that i is injective and continuous.

Let $\eta \in \lambda X$ and W be a neighborhood of φ_η in $S(X)$. We can assume that W consists of all $\varphi \in S(X)$ such that $|\varphi_\eta(f_i) - \varphi(f_i)| < \epsilon_i$ for some functions $f_i \in C(X)$ and $\epsilon_i > 0$, $i = 1, \dots, k$. Let $U_i = \{x \in X : f_i(x) > \varphi_\eta(f_i) - \epsilon_i\}$ and $V_i = \{x \in X : f_i(x) < \varphi_\eta(f_i) + \epsilon_i\}$. Because of (1), for each $i \leq k$ there exist $F_i, H_i \in \eta$ such that $F_i \subset U_i$ and $H_i \subset V_i$. Then $G = \bigcap_{i=1}^k (U_i^+ \cap V_i^+)$ is a neighborhood of η in λX , and $i(\xi) \in W$ for all $\xi \in G$. So, i is continuous.

Suppose $\eta \neq \xi$ are two elements of λX . Then there exist $F_0 \in \eta$ and $H_0 \in \xi$ with $F_0 \cap H_0 = \emptyset$. Let $f \in C_+(X)$ be a function such that $f \leq 1$, $f(F_0) = 1$ and $f(H_0) = 0$. Consequently, $\varphi_\eta(f) = \max_{F \in \eta} \min_{x \in F} f(x) = 1$ and $\varphi_\xi(f) = \min_{H \in \xi} \max_{x \in H} f(x) = 0$. Hence, i is injective.

The second part of this proposition follows directly because $S(X)$ is always connected (as a convex set in $\mathbb{R}^{C(X)}$) while λX is disconnected provided so is X . \square

3. \varkappa -METRIZABLE COMPACTA AND SUPEREXTENSIONS

Recall that a compactum X is openly generated [19] if X is the limit space of an inverse sigma-spectrum with open projections. Shchepin [17] proved that any \varkappa -metrizable compactum is openly generated. On the other hand, every openly generated compactum is \varkappa -metrizable (see [19]) which follows from Ivanov's result [7] that λX is a Dugundji space provided X is openly generated. We are going to use Ivanov's theorem in the proof of Theorem 1.1 next section. That's why we decided to establish in this section an independent proof of a more general fact, see Proposition 3.2. Part of this proof is Lemma 3.1 below. Recall that Lemma 3.1 was established by Shirokov [21] but his proof is based on Ivanov's result mentioned above. Our proof of Lemma 3.1 is based on the Shchepin results [18] about inverse systems with open projections.

We say that a subset X of Y is *regularly embedded* in Y if there exists a correspondence e from the topology of X into the topology of Y such that $e(\emptyset) = \emptyset$, $e(U) \cap X = U$, and $e(U) \cap e(V) = \emptyset$ provided $U \cap V = \emptyset$.

Lemma 3.1. *Let X be an openly generated compactum. Then every embedding of X in a space Y is regular.*

Proof. It suffices to show that every embedding of X in a Tychonoff cube \mathbb{I}^A is regular, where $|A| = w(X)$. This is true if X is metrizable, see [9, §21, XI, Theorem 2]. Suppose the statement holds for any openly generated compactum of weight $< \tau$, and let $X \subset \mathbb{I}^A$ be an openly generated compactum of weight $|A| = \tau$. Then X is the limit space of a well ordered inverse system $S = \{X_\alpha, p_\beta^\alpha, \beta < \alpha < \omega(\tau)\}$ such that all projections $p_\alpha: X \rightarrow X_\alpha$ are open surjections and all X_α are openly generated compacta of weight $< \tau$, see [19]. Here, $\omega(\tau)$ is the first ordinal of cardinality τ . For any $B \subset A$ let π_B be the projection from \mathbb{I}^A to \mathbb{I}^B , and $p_B = \pi_B|X: X \rightarrow \pi_B(X)$. According to Shchepin's spectral theorem [17], we can assume that there exists an increasing transfinite sequence $\{A(\alpha) : \alpha < \omega(\tau)\}$ of subsets of A such

that $A = \bigcup \{A(\alpha) : \alpha < \omega(\tau)\}$, $X_\alpha = p_{A(\alpha)}(X)$ and $p_\alpha = p_{A(\alpha)}$. By [18], X has a base \mathcal{B} consisting of open sets $U \subset X$ with finite rank $d(U)$. Here,

$$d(U) = \{\alpha : p_{\alpha+1}(U) \neq (p_\alpha^{\alpha+1})^{-1}(p_\alpha(U))\}.$$

For every $U \in \mathcal{B}$ let $A(U) = \{\alpha_0, \alpha, \alpha+1 : \alpha \in d(U)\}$, where $\alpha_0 \in A$ is fixed. Obviously, X is a subset of $\prod \{X_\alpha : \alpha < \omega(\tau)\}$. For every $U \in \mathcal{B}$ we consider the open set $\gamma_1(U) \subset \prod \{X_\alpha : \alpha < \omega(\tau)\}$ defined by

$$\gamma_1(U) = \prod \{p_\alpha(U) : \alpha \in A(U)\} \times \prod \{X_\alpha : \alpha \in A \setminus A(U)\}.$$

Now, let

$$\gamma(W) = \bigcup \{\gamma_1(U) : U \in \mathcal{B} \text{ and } \overline{U} \subset W\}, W \in \mathcal{T}_X.$$

Claim. The following conditions hold: (i) $\gamma(W_1) \cap \gamma(W_2) = \emptyset$ whenever W_1 and W_2 are disjoint open sets in X ; (ii) $\gamma(W) \cap X = W$, $W \in \mathcal{T}_X$.

Suppose $W_1 \cap W_2 = \emptyset$. To prove condition (i), it suffices to show that $\gamma_1(U_1) \cap \gamma_1(U_2) = \emptyset$ for any pair $U_1, U_2 \in \mathcal{B}$ with $\overline{U_i} \subset W_i$, $i = 1, 2$. Indeed, we fix such a pair and let $\beta = \max\{A(U_1) \cap A(U_2)\}$. Then β is either α_0 or $\max\{d(U_1) \cap d(U_2)\} + 1$. In both cases $d(U_1) \cap d(U_2) \cap [\beta, \omega(\tau)) = \emptyset$. According to [18, Lemma 3, section 5], $p_\beta(U_1) \cap p_\beta(U_2) = \emptyset$. Since $\beta \in A(U_1) \cap A(U_2)$, $\gamma_1(U_1) \cap \gamma_1(U_2) = \emptyset$.

Obviously, $W \subset \gamma(W) \cap X$ for all $W \in \mathcal{T}_X$. So, condition (ii) will follow as soon as we prove that $\gamma(W) \cap X \subset W$. Fix $x \in \gamma(W) \cap X$ and let $U \in \mathcal{B}$ such that $x \in \gamma_1(U)$ and $\overline{U} \subset W$. Define $\beta(U) = \max d(U) + 1$. Then $p_\alpha(x) \in p_\alpha(U)$ for all $\alpha \leq \beta(U)$. Since $\alpha \notin d(U)$ for all $\alpha \geq \beta(U)$, we have $(p_{\beta(U)}^\alpha)^{-1}(p_{\beta(U)}(x)) \subset p_\alpha(U)$ for $\alpha > \beta(U)$. Hence, $p_\alpha(x) \in p_\alpha(U)$ for all α . The last relation implies that $x \in \overline{U}$, so $x \in W$. This completes the proof of the claim.

We are going to show that $\prod \{X_\alpha : \alpha \in A\}$ is regularly embedded in $\prod \{\mathbb{I}^{A(\alpha)} : \alpha \in A\}$. According to our assumption, each X_α is regularly embedded in $\mathbb{I}^{A(\alpha)}$, so there exists a regular operator $e_\alpha : \mathcal{T}_{X_\alpha} \rightarrow \mathcal{T}_{\mathbb{I}^{A(\alpha)}}$. Let \mathcal{B}_1 be a base for $\prod \{X_\alpha : \alpha \in A\}$ consisting of sets of the form $V = \prod \{U_\alpha : \alpha \in \Gamma(V)\} \times \prod \{X_\alpha : \alpha \in A \setminus \Gamma(V)\}$ with $\Gamma(V) \subset A$ being a finite set. For every $V \in \mathcal{B}_1$ we assign the open set $\theta_1(V) \subset \prod \{\mathbb{I}^{A(\alpha)} : \alpha \in A\}$,

$$\theta_1(V) = \prod \{e_\alpha(U_\alpha) : \alpha \in \Gamma(V)\} \times \prod \{\mathbb{I}^{A(\alpha)} : \alpha \in A \setminus \Gamma(V)\}.$$

Now, we define a regular operator θ between the topologies of $\prod\{X_\alpha : \alpha \in A\}$ and $\prod\{\mathbb{I}^{A(\alpha)} : \alpha \in A\}$ by

$$\theta(G) = \bigcup\{\theta_1(V) : V \in \mathcal{B}_1 \text{ and } V \subset G\}.$$

Since, according to Claim 1, X is regularly embedded in $\prod\{X_\alpha : \alpha \in A\}$ and γ is the corresponding regular operator, the equality $e_1(W) = \theta(\gamma(W))$, $W \in \mathcal{T}_X$, provides a regular operator between the topologies of X and $\prod\{\mathbb{I}^{A(\alpha)} : \alpha \in A\}$. Finally, because $X \subset \mathbb{I}^A \subset \prod\{\mathbb{I}^{A(\alpha)} : \alpha \in A\}$, we define $e: \mathcal{T}_X \rightarrow \mathcal{T}_{\mathbb{I}^A}$,

$$e(W) = \bigcup\{e_1(W) \cap \mathbb{I}^A : W \in \mathcal{T}_X\}.$$

So, X is regularly embedded in \mathbb{I}^A . \square

Proposition 3.2. *Let X be an openly generated compact space possessing a binary closed subbase \mathcal{S} . Then X is a Dugundji space. Moreover, if in addition X is connected and \mathcal{S} is normal, then X is an absolute retract.*

Proof. Suppose X is an openly generated compactum with a binary closed subbase \mathcal{S} and X is embedded in \mathbb{I}^τ for some τ . By Lemma 3.1, X is regularly embedded in \mathbb{I}^τ , so there exists a regular operator $e: \mathcal{T}_X \rightarrow \mathcal{T}_{\mathbb{I}^\tau}$. Define the set-valued map $r: \mathbb{I}^\tau \rightarrow X$ by

$$(2) \quad r(y) = \bigcap\{I_{\mathcal{S}}(\overline{U}) : y \in e(U), U \in \mathcal{T}_X\} \text{ if } y \in \bigcup\{e(U) : U \in \mathcal{T}_X\}$$

$$\text{and } r(y) = X \text{ otherwise,}$$

where \overline{U} is the closure of U in X and $I_{\mathcal{S}}(\overline{U}) = \bigcap\{S \in \mathcal{S} : \overline{U} \subset S\}$. Since e is regular, for every $y \in \mathbb{I}^\tau$ the system $\gamma_y = \{U \in \mathcal{T}_X : y \in e(U)\}$ is linked. Consequently, $\omega_y = \{S \in \mathcal{S} : \overline{U} \subset S \text{ for some } U \in \gamma_y\}$ is also linked, so $r(y) = \bigcap\{S : S \in \omega_y\} \neq \emptyset$.

It is easily seen that $r(x) = x$ for all $x \in X$. Moreover, r is usc. Indeed, let $r(y) \subset W$ with $y \in \mathbb{I}^\tau$ and $W \in \mathcal{T}_X$. Then there exist finitely many $U_i \in \mathcal{T}_X$, $i = 1, 2, \dots, k$, such that $y \in \bigcap_{i=1}^k e(U_i)$ and $\bigcap_{i=1}^k I_{\mathcal{S}}(\overline{U}_i) \subset W$. Obviously, $r(y') \subset W$ for all $y' \in \bigcap_{i=1}^k e(U_i)$. So, r is an usco retraction from \mathbb{I}^τ onto X . According to [2], X is a Dugundji space.

Suppose now X is connected and \mathcal{S} is a binary normal subbase for X . By [10], any set of the form $I_{\mathcal{S}}(F)$ is \mathcal{S} -convex (a set $A \subset X$ is \mathcal{S} -convex if $I_{\mathcal{S}}(x, y) \subset A$ for all $x, y \in A$). So, each $r(y)$ is an \mathcal{S} -convex set. According to [10, Corollary 1.5.8], all closed \mathcal{S} -convex subsets of X are also connected. Hence, the map r , defined by (2), is connected-valued. Consequently, by [2], X is an absolute extensor in dimension 1, and there exists a map $r_1: \mathbb{I}^\tau \rightarrow \exp X$ with $r_1(x) = \{x\}$ for all

$x \in X$, see [3, Theorem 3.2]. Here, $\exp X$ is the space of all closed subsets of X with the Vietoris topology. On the other hand, since X is normally supercompact, there exists a retraction r_2 from $\exp X$ onto X , see [10, Corollary 1.5.20]. Then the composition $r_2 \circ r_1: \mathbb{I}^\tau \rightarrow X$ is a (single-valued) retraction. So, $X \in AR$. \square

Corollary 3.3. ([7],[8]) *λX is a Dugundji space provided X is an openly generated compactum. If, in addition, X is connected, then λX is an absolute retract.*

Proof. It is easily seen that λ is a continuous functor preserving open maps, see [4]. So, λX is openly generated. Since λX has a binary normal subbase, Proposition 3.2 completes the proof. \square

A closed subset \mathcal{A} of $\exp X$ is called an *inclusion hyperspace* if for every $A \in \mathcal{A}$ and every $B \in \exp X$ the inclusion $A \subset B$ implies $B \in \mathcal{A}$. The space of all inclusion hyperspaces with the inherited topology from $\exp^2 X$ is closed in $\exp^2 X$ and it is denoted by GX [11].

Corollary 3.4. [11] *GX is a Dugundji space provided X is an openly generated compactum. If, in addition, X is connected, then GX is an absolute retract.*

Proof. Since G is a continuous functor preserving open maps [22], GX is openly generated if X is so. On the other hand, GX has a binary subbase [13]. Hence, by Proposition 3.2, GX is a Dugundji space. Suppose X is connected and openly generated. Then GX is a connected Dugundji space. Hence, $\lambda(GX)$ is an absolute retract. Moreover, there exist natural inclusions $GX \subset \lambda(GX) \subset G^2X = G(GX)$, and a retraction from G^2X onto GX , see [13, Lemma 2]. Consequently, $GX \in AR$ as a retract of $\lambda(GX)$. \square

4. PROOF OF THEOREM 1.1

We begin this section with the following lemma.

Lemma 4.1. *Let X be a compact regularly embedded subset of a space Y . Then there exists an usco map $r: Y \rightarrow \lambda X$ such that $r(x) = \eta_x$ for all $x \in X$.*

Proof. Let $e: \mathcal{T}_X \rightarrow \mathcal{T}_Y$ be a regular operator between the topologies of X and Y . Define $r: Y \rightarrow \lambda X$ by

$$r(y) = \bigcap \{(\overline{U})^+ : y \in e(U), U \in \mathcal{T}_X\} \text{ if } y \in \bigcup \{e(U) : U \in \mathcal{T}_X\},$$

$$\text{and } r(y) = \lambda X \text{ otherwise ,}$$

where \overline{U} is the closure of U in X and $(\overline{U})^+ \subset \lambda X$ consists of all $\eta \in \lambda X$ with $\overline{U} \in \eta$. Since e is regular, for every $y \in Y$ the system $\{U \in \mathcal{T}_X : y \in e(U)\}$ is linked. Consequently, $\{\overline{U} : y \in e(U)\}$ is also linked and consists of closed sets in X . Hence, $\bigcap \{(\overline{U})^+ : y \in e(U), U \in \mathcal{T}_X\} \neq \emptyset$. So, r has non-empty compact values. It is easily seen that $r(x) = \eta_x$ for all $x \in X$. One can show (as in the proof of Proposition 3.2) that r is usc. \square

Proof of Theorem 1.1 (i) \Rightarrow (ii) We follow the proof of [16, Theorem 3]. The equality (1) provides a monotone homogeneous and weakly additive extender $u_1: C(X) \rightarrow C(\lambda X)$. Consider the space Z obtained by attaching λX to Y at the points of X . Then λX is a closed subset of Z and, since λX is a Dugundji space (see Corollary 3.3), there exists a linear monotone extension operator $u_2: C(\lambda X) \rightarrow C(Z)$ with $u_2(1) = 1$. Then $u(f) = u_2(u_1(f))|_Y$, $f \in C(X)$, defines a monotone homogeneous and weakly additive extender $u: C(X) \rightarrow C(Y)$.

(ii) \Rightarrow (iii) Suppose X is a subset of Y and $u: C(X) \rightarrow C(Y)$ is a monotone homogeneous and weakly additive extender. Define $r: Y \rightarrow S(X)$ by $r(y)(f) = u(f)(y)$, where $y \in Y$ and $f \in C(X)$. It is easily seen that r is continuous and $r(x) = \delta_x$ for all $x \in X$.

(iii) \Rightarrow (iv) Let $r_2: Y \rightarrow S(X)$ be a continuous map with $r_2(x) = \delta_x$ for all $x \in X$. The equality $u(f)(\varphi) = \varphi(f)$ defines a monotone extender $u: C(X) \rightarrow C(S(X))$. Then, by [1, Theorem 4.1], X is regularly embedded in $S(X)$. So, according to Lemma 4.1, there exists an usco map $r_1: S(X) \rightarrow \lambda X$ such that $r_1(\delta_x) = \eta_x$, $x \in X$. Then $r = r_1 \circ r_2: Y \rightarrow \lambda X$ is the required map.

(iv) \Rightarrow (i) Let X be embedded in \mathbb{I}^A for some A and $r: \mathbb{I}^A \rightarrow \lambda X$ be an usco map with $r(x) = \eta_x$, $x \in X$. For every open $U \subset X$ define $e(U) = \{y \in \mathbb{I}^A : r(y) \subset U^+\}$. Since U^+ is open in λX and r is usc, $e(U)$ is open in \mathbb{I}^A . Moreover, $e(U) \cap X = U$ and $e(U) \cap e(V) = \emptyset$ provided U and V are disjoint. So, X is regularly embedded in \mathbb{I}^A . Finally, by [21], X is κ -metrizable. \square

Now, we consider the functor O of order preserving functionals introduced by Radul [14]. Recall that a map $\varphi: C(X) \rightarrow \mathbb{R}$ is called order preserving if it is monotone, weakly additive and $\varphi(1_X) = 1$. Theorem 4.2 below follows from Theorem 1.1. Indeed, since S is a subfunctor of O we have embeddings $X \subset S(X) \subset O(X)$ for every compactum X . This implies implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) of Theorem 4.2. The proofs of implications (iii) \Rightarrow (vi) and (vi) \Rightarrow (i) are the same.

Theorem 4.2. *For a compact space X the following conditions are equivalent:*

- (i) X is \varkappa -metrizable;
- (ii) For any embedding of X in a compactum Y there exists a monotone and weakly additive extender $u: C(X) \rightarrow C(Y)$ such that $u(1_X) = 1_Y$;
- (iii) For any embedding of X in a compactum Y there exists a continuous map $r: Y \rightarrow O(X)$ such that $r(x) = \delta_x$ for all $x \in X$.
- (iv) For any embedding of X in a compactum Y there exists an usco map $r: Y \rightarrow \lambda X$ such that $r(x) = \eta_x$ for all $x \in X$;

Let F be a covariant functor in the category of compact spaces and continuous maps. A compactum X is said to be F -injective [19] if for any map $f: Y \rightarrow X$ and any embedding $i: Y \rightarrow Z$ there exists a map $g: F(Z) \rightarrow F(X)$ such that $F(f) = g \circ F(i)$. It is easily seen that if F has the property that for every X there exists an embedding $j_X: X \rightarrow F(X)$, then X is F -injective iff for every embedding of X in another compactum Y there exists a map $h: Y \rightarrow F(X)$ with $h(x) = j_X(x)$ for all $x \in X$. It follows from Theorem 1.1(iii) that \varkappa -metrizable compacta are exactly the S -injective compacta.

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